Assignment 3

① let $X: U \subseteq \mathbb{R}^2 \to M_1$ be any parametrization of M_1 around p and $X(u_0, v_0) = p$.

$$\begin{tikzcd} M_1 \arrow[r, "F"] & M_2 \\ U \arrow[u, "X"] & \end{tikzcd}$$

For any $\vec{v} = aX_u + bX_v \in T_pM_1$, let $\alpha(t)$ be a smooth curve on M_1 such that

$$\alpha(0) = p, \quad \alpha'(0) = \vec{v}.$$

let $\beta(t) = X^{-1} \circ \alpha(t)$ be a smooth curve on $U \subseteq \mathbb{R}^2$, then

$$\alpha(t) = X \circ X^{-1} \circ \alpha(t) = X \circ \beta(t)$$

$$\alpha'(0) = [X_u, X_v] \cdot \vec{\beta}'(0)^T$$

$\therefore$ we have $\beta'(0) = (a, b)$ since $\alpha'(0) = aX_u + bX_v$.

By definition,

$$dF_p(\vec{v}) = \frac{d}{dt} F(\alpha(t)) \Big|_{t=0}$$

$$= \frac{d}{dt}(F \circ X)(X^{-1} \circ \alpha) \Big|_{t=0}$$ where $F \circ X: U \subseteq \mathbb{R}^2 \to M_2 \subseteq \mathbb{R}^3$ is a smooth map.

$$= D[F \circ X] \cdot \vec{\beta}'(0)$$

$$= D_u(F \circ X) \cdot a + D_v(F \circ X) \cdot b$$

$\therefore dF_p$ does not depend on the choice of $\alpha(t)$.

$\therefore dF_p : T_pM_1 \to T_qM_2$ is well-defined.

- For any $V_1 = aX_u + bX_v$, $V_2 = cX_u + dX_v$

$$\begin{aligned} dF_p(V_1 + V_2) &= dF_p((a+c)X_u + (b+d)X_v) \\ &= D_u(F \circ X) \cdot (a+c) + D_v(F \circ X)(b+d) \\ &= [D_u(F \circ X) \cdot a + D_v(F \circ X) \cdot b] + [D_u(F \circ X) \cdot c + D_v(F \circ X) \cdot d] \\ &= dF_p(V_1) + dF_p(V_2) \end{aligned}$$

$\therefore dF_p$ is linear.

② $X(u,v) = (\phi(v)\cos u,\ \phi(v)\sin u,\ \psi(v))$ where $[\phi'(v)]^2+[\psi'(v)]^2=1$ since v is the arc-length of α.

$\therefore\ X_u = (-\phi\sin u,\ \phi\cos u,\ 0)$

$X_v = (\phi'\cos u,\ \phi'\sin u,\ \psi')$

$\therefore\ N = \dfrac{X_u\times X_v}{|X_u\times X_v|} = \dfrac{(\phi\psi'\cos u,\ \phi\psi\sin u,\ -\phi\phi')}{|(\phi\psi'\cos u,\ \phi\psi\sin u,\ -\phi\phi')|}$ $\quad \phi(v)\neq 0$, otherwise $X_u=0$.

$= \dfrac{(\psi'\cos u,\ \psi'\sin u,\ -\phi')}{\sqrt{(\phi')^2+(\psi')^2}} = (\psi'\cos u,\ \psi'\sin u,\ -\phi')$

• $X_{uu} = (-\phi\cos u,\ -\phi\sin u,\ 0)$

$X_{uv} = (-\phi'\sin u,\ \phi'\cos u,\ 0)$

$X_{vv} = (\phi''\cos u,\ \phi''\sin u,\ \psi'')$

$\therefore\ g_{ij} = \begin{bmatrix} \phi^2 & 0 \\ 0 & (\phi')^2+(\psi')^2 \end{bmatrix} = \begin{bmatrix} \phi^2 & 0 \\ 0 & 1 \end{bmatrix}$

$h_{ij} = \begin{bmatrix} -\phi\psi' & 0 \\ 0 & \phi''\psi' - \phi'\psi'' \end{bmatrix}$

$\therefore$ the curvatures are

$$K = \frac{\det(h)}{\det(g)} = \frac{\phi\phi'\psi'\psi'' - \phi\phi''(\psi')^2}{\phi^2}$$

$$= \frac{\phi\phi'(-\phi'\phi'') - \phi\phi''[1-(\phi')^2]}{\phi^2}$$

$$= \frac{-\phi(\phi')^2\phi'' - \phi\phi'' + \phi(\phi')^2\phi''}{\phi^2}$$

$$= -\frac{\phi''}{\phi}$$

$$H = tr(g^{ij}h_{ij}) = -\frac{\phi\psi'}{\phi^2} + \phi''\psi' - \phi'\psi''$$

$$= \frac{\phi\phi''\psi' - \psi' - \phi\phi'\psi''}{\phi}$$

③ The equation of the surface is

$$X(t,s) = (\sin t \cdot \cos s,\ \sin t \cdot \sin s,\ \cos t + \lg \tan\tfrac{t}{2})$$

- $X_t = (\cos t \cdot \cos s,\ \cos t \cdot \sin s,\ -\sin t + \frac{1}{\sin t})$

  $X_s = (-\sin t \cdot \sin s,\ \sin t \cdot \cos s,\ 0\ )$

- $\vec{N} = \dfrac{X_t \times X_s}{|X_t \times X_s|} = (-\cos t \cdot \cos s,\ -\cos t \cdot \sin s,\ \sin t)$

- $X_{tt} = (-\sin t \cdot \cos s,\ -\sin t \cdot \sin s,\ -\cos t - \frac{\cos t}{\sin^2 t})$

  $X_{ts} = (-\cos t \cdot \sin s,\ \cos t \cdot \cos s,\ 0\ )$

  $X_{ss} = (-\sin t \cdot \cos s,\ -\sin t \cdot \sin s,\ 0\ )$

- $g = \begin{bmatrix} \frac{\cos^2 t}{\sin^2 t} & 0 \\ 0 & \sin^2 t \end{bmatrix}$

- $h = \begin{bmatrix} -\frac{\cos t}{\sin t} & 0 \\ 0 & \sin t \cdot \cos t \end{bmatrix}$

. The Gaussian curvature is

$$K = \frac{\det(h)}{\det(g)} = \frac{-\cos^2 t}{\cos^2 t} = -1$$

④ • $X_x = (1, 0, 2x)$, $X_y = (0, 1, 2ky)$

$\therefore T_pM = \text{span}\{(1,0,0), (0,1,0)\}$ when $p = (0,0,0)$

• $X_{xx} = (0,0,2)$, $X_{xy} = (0,0,0)$, $X_{yy} = (0,0,2k)$

$\because N(p) = (0,0,1)$

$\therefore$ $g_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $h_p = \begin{bmatrix} 2 & 0 \\ 0 & 2k \end{bmatrix}$

$\therefore$ $S_p = g^{-1} \cdot h = \begin{bmatrix} 2 & 0 \\ 0 & 2k \end{bmatrix}$

$\therefore$ $S_p(e_1) = 2e_1$, $S_p(e_2) = 2k \cdot e_2$

the principal curvatures of M at p are

2, $2k$

⑤ Let α be a smooth curve on M parametrized by arc-length such that

$$\alpha(0)=p, \quad \alpha'(0)=\vec{v}=\cos\theta\cdot e_1+\sin\theta\cdot e_2$$

$$\therefore \ k_n=\langle \alpha''(t), N(\alpha(t))\rangle$$

$$=-\langle \alpha'(t), \tfrac{d}{dt}N(\alpha(t))\rangle \quad \text{since } \alpha'(t)\in TM$$

$$=\langle \alpha'(t), S_p(\alpha'(t))\rangle$$

$$\therefore \ k_n(p)=\langle \alpha'(0), S_p(\alpha'(0))\rangle$$

$$=\langle \cos\theta\cdot e_1+\sin\theta\cdot e_2, \ \cos\theta\cdot k_1\cdot e_1+\sin\theta\cdot k_2 e_2\rangle$$

$$=k_1\cos^2\theta+k_2\sin^2\theta.$$

⑥ (i) • Let $\{v_1, v_2\}$ be an orthonormal basis of $T_p(M)$ with $v_1\times v_2=N$

[claim: the Gaussian curvature is given by

$$K=\langle \partial_{v_1}N\times\partial_{v_2}N, N\rangle$$]

Let $X: U\to M$ be any parametrization of M around p such that

$$N=\frac{\partial_u X\times\partial_v X}{|\partial_u X\times\partial_v X|}$$

Let g, h be the respect first and second fundamental forms.

Then $\begin{bmatrix} S_p(X_u)\\ S_p(X_v)\end{bmatrix}=g^{-1}\cdot h\begin{bmatrix} X_u\\ X_v\end{bmatrix}$

and $K=\det(g^{-1}\cdot h)$

let $\begin{bmatrix} v_1\\ v_2\end{bmatrix}=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} X_u\\ X_v\end{bmatrix}$ for some real numbers a,b,c,d

then $\begin{bmatrix} \partial_{v_1}N\\ \partial_{v_2}N\end{bmatrix}=\begin{bmatrix} -S_p(v_1)\\ -S_p(v_2)\end{bmatrix}=-\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} S_p(X_u)\\ S_p(X_v)\end{bmatrix}=-\begin{bmatrix} a & b\\ c & d\end{bmatrix}g^{-1}\cdot h\begin{bmatrix} X_u\\ X_v\end{bmatrix}$

$$\therefore \ \partial_{v_1}N\times\partial_{v_2}N=(ad-bc)\cdot\det(g^{-1}\cdot h)\cdot X_u\times X_v$$

$$=(ad-bc)\cdot K\cdot\frac{1}{ad-bc}\, v_1\times v_2 \quad \text{since } \begin{bmatrix} v_1\\ v_2\end{bmatrix}=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} X_u\\ X_v\end{bmatrix}$$

$$=K\cdot v_1\times v_2$$

$$=K\cdot N$$

$$\therefore \ K=\langle K\cdot N, N\rangle=\langle \partial_{v_1}N\times\partial_{v_2}N, N\rangle$$

• $\because \ d(fN)(\vec{v})=\partial_{\vec{v}}f\cdot N+f\cdot\partial_{\vec{v}}N$

$$\therefore \ d(f\vec{N})(\vec{v_1})\times d(f\vec{N})(\vec{v_2})$$

$$=[\partial_{v_1}f\cdot N+f\cdot\partial_{v_1}N]\times[\partial_{v_2}f\cdot N+f\cdot\partial_{v_2}N]$$

$$= -f\partial_{v_2}f \cdot N\times\partial_{v_1}N + f\partial_{v_1}f\cdot N\times\partial_{v_2}N + f^2\partial_{v_1}N\times\partial_{v_2}N$$

$\therefore \langle d(fN)(v_1)\times d(fN)(v_2), N\rangle$

$= f^2\langle \partial_{v_1}N\times\partial_{v_2}N, N\rangle$ since $\langle \partial_{v_1}N\times N, N\rangle = 0$

$\therefore K = \langle \partial_{v_1}N\times\partial_{v_2}N, N\rangle = \dfrac{1}{f^2}\langle d(fN)(v_1)\times d(fN)(v_2), N\rangle.$

(ii) $\because M = h^{-1}(1)$

$\therefore \dfrac{\nabla h}{|\nabla h|}$ is a normal vector field of M.

$\because |\nabla h| = \left|\left(\dfrac{2x}{a^2}, \dfrac{2y}{b^2}, \dfrac{2z}{c^2}\right)\right|$

$= 2\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}} = 2f$

$\therefore fN = f\cdot\dfrac{\nabla h}{|\nabla h|} = f\cdot\dfrac{\nabla h}{2f} = \left(\dfrac{x}{a^2}, \dfrac{y}{b^2}, \dfrac{z}{c^2}\right)$

$\therefore$ if $V = (x_1, y_1, z_1)$, then

$d(fN)(v) = d\left(\dfrac{x}{a^2}, \dfrac{y}{b^2}, \dfrac{z}{c^2}\right)(v) = \left(\dfrac{x_1}{a^2}, \dfrac{y_1}{b^2}, \dfrac{z_1}{c^2}\right)$

Let $\{v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2)\}$ be an orthonormal basis of TM such that $v_1\times v_2 = N$

$\therefore (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1) = N = \dfrac{\nabla h}{|\nabla h|}$

$= \dfrac{\nabla h}{2f} = \dfrac{1}{f}\left(\dfrac{x}{a^2}, \dfrac{y}{b^2}, \dfrac{z}{c^2}\right)$

$\therefore$ By (i), $K = \dfrac{\langle d(fN)(v_1)\times d(fN)(v_2), N\rangle}{f^2}$

$= \dfrac{\left\langle\left(\frac{x_1}{a^2}, \frac{y_1}{b^2}, \frac{z_1}{c^2}\right)\times\left(\frac{x_2}{a^2}, \frac{y_2}{b^2}, \frac{z_2}{c^2}\right), (y_1z_2-y_2z_1, x_2z_1-x_1z_2, x_1y_2-x_2y_1)\right\rangle}{f^2}$

$= \dfrac{\frac{(y_1z_2-y_2z_1)^2}{b^2c^2} + \frac{(x_1z_2-x_2z_1)^2}{a^2c^2} + \frac{(x_1y_2-x_2y_1)^2}{a^2b^2}}{f^2}$

$= \dfrac{1}{f^2}\left(\dfrac{1}{b^2c^2}\cdot\dfrac{x^2}{f^2a^4} + \dfrac{1}{a^2c^2}\cdot\dfrac{y^2}{f^2b^4} + \dfrac{1}{a^2b^2}\cdot\dfrac{z^2}{f^2c^4}\right)$

$= \dfrac{1}{f^4a^2b^2c^2}\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)$

$= \dfrac{1}{f^4a^2b^2c^2}$ since (x,y,z) lies on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1.$